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ON Φ-PSEUDO-VALUATION RINGS II

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ABSTRACT. A commutative ring R with identity $1 \neq 0$ is called a pseudo-valuation ring (PVR) if for every $a, b \in R$, either $a \mid b$ in R or $b|ac$ in R for every nonunit c of R. Also, R is called a Φ-pseudo-valuation ring if $Nil(R)$ (the set of nilpotent elements of R) is a divided prime ideal of R and for every $a, b \in R \backslash Nil(R)$, either a divides b in R or b divides ac for every nonunit c of R. In this paper, we will show that for any $n \geq 0$ (possibly infinite) there is a Φ-PVR of Krull dimension n that is not a PVR.

1. INTRODUCTION

We assume throughout that all rings are commutative with $1 \neq 0$. we begin by recalling some back ground material. As in [9], an integral domain R, with quotient field K, is called a pseudo-valuation domain (PVD) in case each prime ideal P of R is strongly prime, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. In [4], Anderson, Dobbs and the author generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nozero zerodivisors). Recall from [4] that a prime ideal of R is said to be strongly prime (in R) if aP and bR are comparable for all $a, b \in R$. A ring R is called a pseudo-valuation ring (PVR), see Proposition 1.1(5). A PVR is necessarily quasilocal [[4], Lemma 1(b)]; a chained ring is a PVR if and only if it is a PVD; a chained ring is a PVR [[4], Corollary 4]; an integral domain is a PVR if and only if it is a PVD (cf. [[1], Proposition 3.1], [[2], Proposition 4.2], and [[6], Proposition 3]). Recall from [8] and [7] that a prime ideal P of R is called divided if it is comparable to every ideal of R. A ring R is called a divided ring if every prime ideal of R is divided. In [5], the author gave another generalization of PVDs to the context of arbitrary rings (possibly with nonzero zerodivisors).

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$b \nmid a^2$ (in D) by Proposition 2.5(2). Thus, $a \mid b$ (in D). Hence, $(0,1) = (c,d)(x,1)$ for some $(c,d) \in D$. Thus, $xc = 0$ in R and $xd + c = 1$ in B. Since dx is a nonunit of B and B is quasilocal, $c = 1 - xd$ is a unit of B. Since $c^2 = c - xcd = c - 0 = c$ in B and c is a unit of B and 1 is the only unit of B that is an idempotent of B, $c = 1$ in B. Hence, for some $z \in R \setminus Nil(R)$ we have $z(c-1) = 0$ in R. Since $Nil(R)$ is prime and $z \in R \setminus Nil(R)$, $c - 1 = w \in Nil(R)$. Hence, $c = 1 + w$ is a unit of R. A contradiction, since $x \neq 0$ in R and $xc = 0$. Hence, $a \nmid b$ in D. Thus, our denial is invalid. Thus, D is not a PVR. Now we show that D is a Φ−PVR. First, recall that $Nil(D) = Nil(R)(+)B$ by Proposition 2.5(3), and $Nil(D)$ is prime by Proposition 1.2(1). Let $a := (c,d) \in D \setminus Nil(D)$, and $b := (x,y) \in Nil(D)$. Then, $c \in R \setminus Nil(R)$ and $x \in Nil(R)$. Since R is divided by Proposition 1.1, $c \mid x$ in R. Hence, $a \mid b$ in D by Proposition 2.5(3). Thus, $Nil(D)$ is a divided prime ideal of D. Now, let $a := (c,d) \in D \setminus Nil(D)$, $b := (x,y) \in D \setminus Nil(D)$, and $f := (m,g)$ be a nonunit of D. Then m is a nonunit of R by Proposition 1.2(3) and $c, x \in R/Nil(R)$. Suppose that $a \nmid b$ in D. Then $c \nmid x$ in R by Proposition2.5(1). Hence, $x \mid cm$ by Proposition 1.1(6). Thus, $b \mid af$ in D by Proposition 2.5(2). Hence, R is a Φ−PVR by Proposition 1.1(6). By Proposition 1.2(1), $\dim(D) = \dim(R) = n$. □

In light of the proof of the above Theorem, we have the following corollaries.

Corollary 2.7. *Let $d \geq 2$, and let R be a PVR of Krull dimension $n \geq 0$ such that $x^d \neq 0$ in R for some $x \in Nil(R)$, and let $B := R_{Nil(R)}$ as an R-module. Then $D := R(+)B$ is a Φ−PVR of Krull dimension n that is not a PVR, and $y^d \neq 0$ in D for some $y \in Nil(D)$.*

Corollary 2.8. *Let $n \geq 2$ and $d \geq 2$. Then there is a Φ−PVR D with maximal ideal M and Krull dimension n that is not a PVR such that $Z(D)$ is properly contained between $Nil(D)$ and M, and $x^d \neq 0$ in D for some $x \in Nil(D)$.*

Proof. Let R be as in Corollary 2.3 and D as in Theorem 2.6. □

Corollary 2.9. *Let $d \geq 2$. Then there is a Φ−PVR D with maximal ideal M and infinite Krull dimension that is not a PVR such that $Z(D)$ is properly contained between $Nil(D)$ and M, and $x^d \neq 0$ in D for some $x \in Nil(D)$.*

Proof. Let R be as in Proposition 2.4 and D as in Theorem 2.6. □

3. Zero dimensional Φ− PVRs and PVRs

We start with the following proposition.

Proposition 3.1. *Let R be a quasilocal ring with maximal ideal M, and $B := R/M$ as an R-module. Set $D := R(+)B$. Then*

1. $Z(D) := M(+)B$.
2. *D is a chained ring if and only if R is a field.*
3. *D is a PVR if and only if $M^2 = 0$ in R.*

PROOF. 1. This is clear by Proposition 1.2(2).

2. If R is a field, then it is easy to see that D is a chained ring. Hence, assume that R is not a field. Let x be a nonzero element in M. Then neither of $(x, 1)$ and $(0, 1)$ divides the other in D. Hence, D is not a chained ring.
3. Suppose that D is a PVR. Let $a := (x, 1) \in D$ and $b := (y, 1) \in D$ for some $x \in R$. Then $a \nmid b$ in D by the same argument as in (1). Hence, $b \mid ac$ for each nonunit c of D by Proposition 1.1 (5). Thus, $0 \mid xy$ in R for each nonunit y of R. Hence, $xy = 0$ in R for each nonunit y and x of R. Thus, $M^2 = 0$ in R. Now, suppose that $M^2 = 0$ in R. Let a, b be nonunit elements of D and assume that $a \nmid b$ in D. Let c be a nonunit element of D. Then $ac = (0, 0)$ since $M^2 = 0$ in R. Thus, $b \mid ac$. Hence, D is a PVR by Proposition 1.1(5).

□

Proposition 3.2. *Let H be a field. Then there is a PVR D with maximal ideal M that is not a chained ring such that $D/M \approx H$, and there is a Φ−PVR F with maximal ideal N that is not a PVR such that $F/N \approx H$.*

PROOF. Consider $R := H[x]/(x^2)$, $W := H[x]/(x^3)$, and $B := H[x]/(x)$ as an R−module and W−module. Set $D := R(+)B$ and $F := W(+)B$. Then D is a PVR with maximal ideal $M := (x)/(x^2)(+)B$ by Proposition 3.1(3), and it is not a chained ring by Proposition 3.1(2). It is easy to see that $D/M \approx H$. Now, since $Nil(F) = (x)/(x^3)(+)B$ is the maximal ideal of F, F is a Φ−PVR by proposition 1.1(6). Also, F is not a PVR by Proposition 3.1 (3). Once again, it is clear that $F/N \approx H$. □

It is easy to see that Z_n (the ring of integers module n) is a chained ring iff it is a PVR iff it is a Φ−PVR iff $n = P^m$ for some prime $P > 0$ of Z and $m \geq 1$. For finite rings we have the following result.

Proposition 3.3. *Let H be a finite field. Then there is a finite PVR D with maximal ideal M that is not a chained ring such that $D/M \approx H$, and there is a finite Φ−PVR F with maximal ideal N that is not a PVR such that $F/N \approx H$.*

PROOF. Let R, W, B, D, F as in the proof of Proposition 3.2. Then D and F are the desired rings. □

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